Exercise 4

1. Find the interior points and boundary points of the following sets:

   (a) $E_1 = \{(x, y) : \ x \in [0, a], y \in [0, b]\}$.

   (b) $E_2 = \{(x, y, z) : \ z > x^2 + y^2 - 1\}$.

   (c) $E_3 = \{(x, y, z) : \ 1 < x^2 + y^2 + z^2 \leq 4\}$.

   (d) $E_4 = \{(x, y) : \ x \in [0, 1]\}$.

   (e) $E_5 = \{(x, y) : x, y \in \mathbb{Z}\}$.

   (f) $E_6 = \left\{(x, y) : \ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1\right\}, \quad a, b > 0$.

   The determine whether these sets are open, closed, or compact. A set is compact if it is closed and bounded.

2. Let A and B be open sets in $\mathbb{R}^n$. Show that

   (a) $A \bigcup B$ is open.

   (b) $A \bigcap B$ is open.

3. (a) Show that $(a, b), -\infty \leq a < b \leq \infty$, is open.

   (b) Show that $[a, b], -\infty < a \leq b < \infty$, is open. Note that it implies that the singleton set $\{a\}$ is closed.

4. * Prove that F is a closed set in $\mathbb{R}^n$ if and only if every convergent sequence in F has its limit in F.

5. * Prove that whenever F is a closed set containing E, then it must also contain $\overline{E}$. It shows that the closure of a set is the smallest closed set containing this set.

6. Study the limit of the following functions at $(0, 0)$.

   (a)
   $$f(x, y) = \frac{x^2 y^2}{|x| + y^2} \ .$$

   (b)
   $$g(x, y) = \frac{\sin xy}{x^2 + y^2} \ .$$

   (c)
   $$h(x, y) = y \log(x^2 + |y|) \ .$$

   Hint: In (c) examine the sets $\{(x, y) : y \geq x^2\}$ and $\{(x, y) : \ y < x^2\}$ separately.

7. Find the iterated limits and limit of the function
   $$h(x, y) = \frac{x - y}{x + y}$$
   at $(0, 0)$.

8. Consider the function
$$F(x,y) = \frac{x^2y^2}{x^2y^2 + (x-y)^2} \ .$$
Show that
$$\lim_{y\to 0}\lim_{x\to 0} F(x,y) = \lim_{x\to 0}\lim_{y\to 0} F(x,y) = 0 \ ,$$
but
$$\lim_{(x,y)\to(0,0)} F(x,y)$$
does not exist.

9. Describe the natural domains of the functions determined by the following formulas and then study the continuity of these functions.
   (a) $\dfrac{1}{x^2+y^2-1}$,
   (b) $\log(y-x^2)$,
   (c) $\arcsin \dfrac{x}{y}$,
   (d) $\exp\left(\dfrac{-1}{x^2+y^2+z^2}\right)$.

   Here arcsin is the branch of the inverse of the sine function from $[-1,1]$ to $[-\pi/2, \pi/2]$.

10. Use Theorem 4.10 to determine whether the following sets are open or closed:
    (a) $S_1 = \{x \in \mathbb{R}^n : \ p(x) = 0 \ \}$ where p is a polynomial.
    (b) $S_2 = \{(x,y) \in \mathbb{R}^2 : \cos x^2 - \sin^3 xy \le 1\}$.
    (c) $S_3 = \{(x,y,z) : \ x^2 + y^2 < \sin(x+z) < 28z^2\}$.